General Setting

$$\dot{q} = \underbrace{f_u(q)}_{\text{Smooth in } (u,q)} \qquad q \in M, \quad u \in U \qquad \begin{matrix} \dim M = 2 \\ U = \mathbb{R} \text{ or } S^1 \end{matrix} \tag{23.1}$$

$f_u(q) \in T_qM$ — An ODE

ex). $\dot{q} = \cos u\, f_1(q) + \sin u\, f_2(q) \qquad U = S^1$

$\{f_1, f_2\}$: Orthonormal frame of M

Goal).
1. Obtain feedback-invariant curvature κ
2. example of this curvature $\approx$ gaussian curvature.
3. Obtain Jacobi equation

Assumption) Regularity Condition

Curve of admissible Velocity of (23.1) Satisfies

$$\begin{cases} f_u(q) \wedge \dfrac{\partial f_u(q)}{\partial u} \neq 0 \\ \dfrac{\partial f_u(q)}{\partial u} \wedge \dfrac{\partial^2 f_u(q)}{\partial u^2} \neq 0 \qquad \forall q \in M,\ u \in U \end{cases} \tag{23.2}$$

This implies $f_u(q), \dfrac{\partial f_u}{\partial u}, \dfrac{\partial^2 f_u}{\partial u^2}$: linearly indep.

- meaning).
  - $\{f_u(q) \mid u \in M\} \subseteq T_qM$ is Strongly convex.
  - Strong Legendre Condi for extremals of (23.1).

- Hamiltonian $h_u(\lambda) := \langle \lambda_q, f_u(q) \rangle$ for $\lambda \in T^*M$

  $f_u(q) = \dot{q}$: velocity

$H(\lambda) := \max_u h_u(\lambda)$ $\exists!\ \underbrace{u : \text{maximizer}}$ (assume)

meaning). any line of support touches the curve of admissible velocity at unique point

$\underset{(23.2)}{\Rightarrow}$ $H(\lambda)$: Smooth in this domain

H : Homogeneous of Order = 1 on fibers.

$\mathcal{H} := H^{-1}(\{1\}) \subseteq T^*M$: level Surface $\because$ 3-dim $(4-1)$ $M \times \mathbb{R}^2$

$\mathcal{H}_q := \mathcal{H} \cap \underbrace{T_q^*M}_{2\text{-dim}}$: fiber of $\mathcal{H}$ $\to$ dim 1 : Curve $(2-1)$

q: position. p: velocity

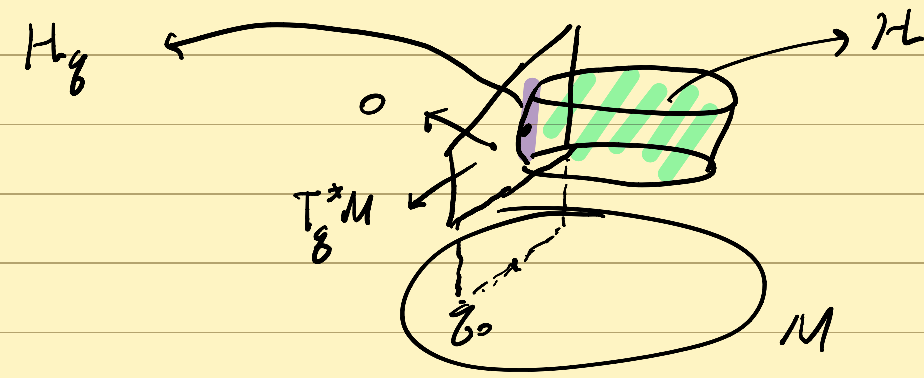
23.1.1. ※ Constructing FB-invariant moving frame

$(\tilde{u}(t), \lambda_t)$: optimal

Goal). Write Jacobi equation : $\dot{\eta} = \vec{b}_t(\eta)$ $\eta \in \Sigma = T_{\lambda_0}(T^*M)$

Rmk). $\max_\lambda H(\lambda)$: depends on only $\dot{q} = f_u(q) \Rightarrow$ invariant under parametrization of u.

$\Rightarrow \mathcal{H}, \mathcal{H}_q$: feedback invariant. invariant on action of feedback



$\mathcal{H}_q$: Curve parametrized by φ ⟶ 1-dim lower as $\mathcal{H}$: level set ⇒ $\mathcal{H}_q$: 1-dim (Curve in $T_q M$)

$\mathcal{H}_q$: Not passing the origin ($\because$ $0 \notin \mathcal{H} \Leftarrow H(0) \neq 1$) $p(\varphi) \neq 0$.

$p(\varphi) \wedge \frac{dp(\varphi)}{d\varphi} \neq 0$ ⟵ (23.2)'s dual ($\because \lambda$: dual of $f_u(q)$. from const of $\mathcal{H}$)

$\Rightarrow \{p(\varphi), \frac{dp(\varphi)}{d\varphi}\}$: frame of T_q^*M ($\because \wedge \neq 0 \Rightarrow$ linear indep, $\dim T_q^*M = 2$)

$\therefore \frac{d^2p}{d\varphi^2} = a_1(\varphi)p(\varphi) + a_2\frac{dp(\varphi)}{d\varphi}$ ($\because \frac{d^2p}{d\varphi^2} \in T_q^*M$)

↓ Strong Convexity of $\mathcal{H}_q = p(\varphi)$

$a_1(\varphi) < 0$

$\theta := \theta(\varphi)$, reparam

$\frac{d^2p}{d\theta^2} = a_1(\varphi)\left(\frac{d\varphi}{d\theta}\right)^2 p(\theta) + \tilde{a}_2(\theta)\frac{dp}{d\theta}(\theta)$

$\therefore \frac{d^2p}{d\theta^2} = -p(\theta) + b(\theta)\frac{dp}{d\theta}(\theta)$, by Setting θ as $\left(\frac{d\varphi}{d\theta}\right)^2 = -a_1(\varphi)$

$\operatorname{Ker} d\pi = \operatorname{span}\{\frac{\partial}{\partial\theta}\}$ $\pi: T^*M \to M$

for this θ, we define $v := \frac{\partial}{\partial\theta}$: Vertical v.f. on $\mathcal{H}$ $\to \simeq \{\theta\} \times M$

$\cap$ T^*M ↖ T_q^*M

where this come?

Lie deri

v: unique Vertical v.f. s.t. $L_v^2 s = -s + bL_v s$ (*) where $s = pdq$: Tautological 1 form restricted $\mathcal{H}$

$\mathcal{H}$'s moving frame : $\{v, [v, \vec{H}], \vec{H}\}$: linearly indep.

$\rotatebox{90}{=}$ V_1, V_2, V_3

↳ below remark

$\vec{H} = \sum_i \left[\frac{\partial H}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial \xi_i}\right]$

$\xi_i \in T_q^*M$, $x_i \in M$

v: Vertical. $[v, \vec{H}]$ & $\vec{H}$: linear indep Horizontal part.

↳ $\{\theta\}$ $M \setminus \vec{H}$ $\{\theta\}^{\perp}_{\mathcal{H}}$

direct.

Rmk). $\dot{q} = f_u(q)$ $q \in M$
$\dot{\lambda} = \vec{H}(\lambda)$ $\lambda \in T^*M$ $\Rightarrow (d\pi)\vec{H} = f_u$, $\underbrace{(d\pi)[v, \vec{H}] = \frac{\partial f_u}{\partial u}\frac{\partial u}{\partial \theta}}$ $\Rightarrow$ indep by (23.2)

$\left(\because [\frac{\partial}{\partial\theta}, \vec{H}](\lambda) = \frac{\partial}{\partial\theta}\vec{H}(\lambda) - \vec{H}(\frac{\partial}{\partial\theta})(\lambda) \right)$
$d\pi$ { $\frac{\partial}{\partial\theta}\dot{q} = 0$ $\quad d\pi \hookrightarrow \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial \theta}$
($\because$ θ: Vertical $\Rightarrow \frac{\partial}{\partial\theta} \in \ker d\pi$)

- For Jacobi eq: We need $\underbrace{[\vec{H}, V_1]}_{= -V_2 \text{ by def}}$, $[\vec{H}, V_2]$, $\underbrace{[\vec{H}, V_3]}_{=0}$ ($\because V_3 = \vec{H}$)

Main Thm
Thm 23.1). $[\vec{H}, -V_2] = [\vec{H}, [\vec{H}, v]] = -kv$
$K(\lambda)$ $\lambda \in \mathcal{H}$ is called a curvature of the system.
$\hookrightarrow$ Feedback invariant as { $\vec{H}$'s invariance comes from $\mathcal{H}$'s invar.
v: invariant

Rmk). v: unique up to sgn $\Rightarrow$ k: does not matter with v's sgn (23.6).

pf). $\mathcal{H} \simeq \{\theta\} \times M$ ($\because$ θ = parameterization of $\varphi \Leftarrow$ parametrization of $\mathcal{H}_q$)
$\therefore \simeq \underbrace{\mathcal{H}_{\cdot}}_{q} \times M$.
$\therefore$ $\mathcal{H}$ can be decomposed Hori & Verti
$\vec{H} = \underbrace{Y}_{\text{Hori}} + \alpha \underbrace{\frac{\partial}{\partial\theta}}_{\text{Vertical}}$ $\quad \alpha = \alpha(\theta, q)$ as $\vec{H} \in Vec(\mathcal{H})$: M: Hori, $\{\theta\}$: Verti

to find α
Set s: Tautological 1-form on $\Lambda^1(\mathcal{H})$ $\in \Lambda^1(\mathcal{H})$
- Claim). $\mathcal{H}$'s dual Coframe $= \{d\theta, s, L_{\frac{\partial}{\partial\theta}} s\}$ s: Tautological 1-form restricted to $\mathcal{H}$
dual with $\{\frac{\partial}{\partial\theta}, Y, [\frac{\partial}{\partial\theta}, Y]\}$
- Claim pf). w.t.s. $\{\frac{\partial}{\partial\theta}, Y, [\frac{\partial}{\partial\theta}, Y]\} \overset{\text{dual}}{\longleftrightarrow} \{d\theta, s, [d\theta, s]\}$
trivial ① ②:
Suffice to show this

①. $Y^* = s$
apply $d\pi$ to linearize (equality because original space was v.s.)
$\langle s_\lambda, \frac{\partial}{\partial\theta}\rangle = \langle \lambda, \underbrace{d\pi(\frac{\partial}{\partial\theta})}_{=0}\rangle = 0$ ($\because$ $\frac{\partial}{\partial\theta}$: Vertical $\in \ker(d\pi)$)

for $\lambda \in \mathcal{H}$
($\because$ s: restricted)
$\langle s_\lambda, Y\rangle = \langle s_\lambda, \vec{H} - \alpha\frac{\partial}{\partial\theta}\rangle = \langle s_\lambda, \vec{H}\rangle - \alpha\langle s_\lambda, \frac{\partial}{\partial\theta}\rangle$ $=0$ from the above
$= \langle \lambda, f_u\rangle$ } linearize the v.s.
def { $= H(\lambda)$
1-level set { $= 1$

$\langle s_\lambda, [\frac{\partial}{\partial\theta}, Y]\rangle = \langle s_\lambda, [\frac{\partial}{\partial\theta}, \vec{H} - \alpha\frac{\partial}{\partial\theta}]\rangle = \langle s_\lambda, [\frac{\partial}{\partial\theta}, \vec{H}]\rangle$
linearize $= \langle \lambda, \frac{\partial f_u}{\partial u}\frac{\partial u}{\partial\theta}\rangle = \underbrace{\langle \lambda, \frac{\partial f_u}{\partial u}\rangle}_{=0} \frac{\partial u}{\partial\theta} = 0$ ($\because = \frac{\partial}{\partial u}\underbrace{\langle\lambda, f_u\rangle}_{H(\lambda)=\text{const}}$)

$\therefore Y^* = s$ $\square$①.

② $\left[\frac{\partial}{\partial\theta}, Y\right]^* = [d\theta, s]$

- $\langle L_{\frac{\partial}{\partial\theta}} s, \frac{\partial}{\partial\theta} \rangle = \langle d\pi(L_{\frac{\partial}{\partial\theta}} s), 0 \rangle = 0$
- $\langle L_{\frac{\partial}{\partial\theta}} s, Y \rangle = [d\theta, \langle s, Y\rangle] - \langle s, [\frac{\partial}{\partial\theta}, Y]\rangle$ ($\because$ Leibniz rule on Lie deri) $= 0$

$\langle L_{d\theta} s, Y\rangle = \underbrace{L_{d\theta}(\overbrace{\langle s, Y\rangle}^{=1 \text{ (from ①)}})}_{=0} - \underbrace{\langle s, [\frac{\partial}{\partial\theta}, Y]\rangle}_{0 \text{ (from ①)}}$

- $\langle s', Y' \rangle = \underbrace{L_{\frac{\partial}{\partial\theta}}(\langle s', Y\rangle)}_{=0 \text{ as } \langle s', Y\rangle = 0} - \langle s'', Y\rangle \underset{(*)}{=} \langle s - bs', Y\rangle = \langle s, Y\rangle = 1.$

$\Rightarrow$ ①.②: $\{d\theta, s, [d\theta, s]\}$: Coframe of $\Lambda^1(\mathcal{H})$ □ (claim)

dual $\longleftrightarrow$ $\{\frac{\partial}{\partial\theta}, Y, [\frac{\partial}{\partial\theta}, Y]\}$.

- under claim). $\{\frac{\partial}{\partial\theta}, Y, [\frac{\partial}{\partial\theta}, Y]\}$: frame of $\mathcal{H}$. ($\because$ dual of Coframe)

$\in Vec(\mathcal{H})$

Now, let $\sigma|_{\mathcal{H}} := d(s|_{\mathcal{H}})$: symplectic form on $\in \Lambda^2(\mathcal{H})$.

$\therefore$ $\sigma|_{\mathcal{H}} = d\theta\wedge s' + \underbrace{d_q s}_{\in\Lambda^2(\mathrm{Hori}(\mathcal{H}))}$ $= d\theta\wedge s' + cs\wedge s'$

$= c\,\underbrace{s\wedge s'}_{\text{frame}}$

$L_{\vec{H}} s = d(i_{\vec{H}} s) + i_{\vec{H}}(ds)$

Now, $i_{\vec{H}}\,\sigma|_{\mathcal{H}} \underset{\text{def of } \sigma \,\&\, \vec{H}}{=} dH|_{\mathcal{H}} = 0$ ($\because$ on $\mathcal{H}$, H is const)

$\therefore$ $i_{\vec{H}}\,\sigma|_{\mathcal{H}} = \sigma|_{\mathcal{H}}(\vec{H}, \cdot) = \sigma|_{\mathcal{H}}(Y + \alpha\frac{\partial}{\partial\theta}, \cdot) = (d\theta\wedge s' + cs\wedge s')(Y + \alpha\frac{\partial}{\partial\theta}, \cdot)$

$= d\theta(Y + \alpha\frac{\partial}{\partial\theta})\wedge s'(\cdot) + cs(Y + \alpha\frac{\partial}{\partial\theta})\wedge s'(\cdot)$

$= \alpha s'(\cdot) + cs'(\cdot)$

$\therefore$ $(\alpha + c)s'(\cdot) = 0 \Rightarrow \alpha = -c$

$\therefore$ $\vec{H} = Y - c\frac{\partial}{\partial\theta}$

Now, able to compute $[\vec{H}, V_2]$

$[\vec{H}, V_2] = [\vec{H}, \underbrace{[\vec{H}, \frac{\partial}{\partial\theta}]}] = [Y - c\frac{\partial}{\partial\theta}, -Y' + c'\frac{\partial}{\partial\theta}]$

$= [Y - c\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}]$

$= -Y' - \left(\underbrace{\frac{\partial}{\partial\theta}(c\frac{\partial}{\partial\theta})}_{\frac{\partial c}{\partial\theta}\cdot\frac{\partial}{\partial\theta} + \cancel{c\frac{\partial^2}{\partial\theta^2}}} - c\frac{\partial}{\partial\theta}(\frac{\partial}{\partial\theta})\right)$ $\quad -\cancel{c\frac{\partial^2}{\partial\theta^2}}$

$= -\left(Y' - [\frac{\partial}{\partial\theta}, c]\frac{\partial}{\partial\theta}\right)$

$= -[Y, -Y'] - [c\frac{\partial}{\partial\theta}, -Y']$

$+ [Y, c'\frac{\partial}{\partial\theta}] + [-c\frac{\partial}{\partial\theta}, c'\frac{\partial}{\partial\theta}]$

$= [Y', Y] + cY'' - c'Y$

$+ \left[\underbrace{-c\frac{\partial}{\partial\theta}(c'\frac{\partial}{\partial\theta})} - c'\frac{\partial}{\partial\theta}(-c\frac{\partial}{\partial\theta})\right]$

$-c\frac{\partial c'}{\partial\theta}\frac{\partial}{\partial\theta} - \cancel{cc'\frac{\partial^2}{\partial\theta^2}}$

$+ c'\frac{\partial c}{\partial\theta}\frac{\partial}{\partial\theta} - \cancel{c'c\frac{\partial^2}{\partial\theta^2}}$

$= c'\frac{\partial c}{\partial\theta} - c\frac{\partial c'}{\partial\theta}$

$= [Y', Y] + cY'' - c'Y + (\vec{H}c' - \vec{H}'c)\frac{\partial}{\partial\theta}$

$[Y - c\frac{\partial}{\partial\theta}, -Y' + c'\frac{\partial}{\partial\theta}] = [Y', Y] + [Y, c'\frac{\partial}{\partial\theta}] + [c\frac{\partial}{\partial\theta}, Y'] - [c\frac{\partial}{\partial\theta}, c'\frac{\partial}{\partial\theta}]$

$[Y, c'\frac{\partial}{\partial\theta}] = Y(c'\frac{\partial}{\partial\theta}) - c'\frac{\partial}{\partial\theta}(Y) = Y(c')\frac{\partial}{\partial\theta} + \underbrace{c' Y\circ\frac{\partial}{\partial\theta} - c'\frac{\partial}{\partial\theta}\circ Y}$

$= c'[Y, \frac{\partial}{\partial\theta}]$

$= -c'Y'$

$[c\frac{\partial}{\partial\theta}, Y'] = c\frac{\partial}{\partial\theta}\circ Y' - Y'\circ c\frac{\partial}{\partial\theta} = c\frac{\partial}{\partial\theta}\circ Y' - Y'(c)\frac{\partial}{\partial\theta} - cY'\circ\frac{\partial}{\partial\theta}$

$= -Y'(c) + c[\frac{\partial}{\partial\theta}, Y']$

$= -Y'(c) + cY''$

$[c\frac{\partial}{\partial\theta}, c'\frac{\partial}{\partial\theta}] = c'\frac{\partial c}{\partial\theta} - c\frac{\partial c'}{\partial\theta}$

Using $\vec{H}c' = Y(c) - c\frac{\partial c'}{\partial\theta}$, $\vec{H}'c = Y'(c) - c'\frac{\partial c}{\partial\theta}$,

$\Rightarrow [\vec{H}, V_2] = \underbrace{[Y', Y] + cY'' - c'Y}_{\text{Horizontal} \cdots \circledast\circledast} + \underbrace{(\vec{H}c' - \vec{H}'c)\frac{\partial}{\partial\theta}}_{\text{Vertical}}$

Recall the goal again: $[\vec{H}, V_2] = -kv$, $v = \frac{\partial}{\partial\theta}$

$\therefore$ Need to show $\circledast\circledast = 0$

From $(*)$, $s'' = -s - bs' \overset{\text{dual}}{\Leftrightarrow} Y'' = -Y - bY'$

$\therefore cY'' - c'Y' = -cY - bcY' - c'Y' = -cY - (bc + c')Y'$

and using Prop 18.3 : $Y \overset{\text{dual}}{\leftrightarrow} s$, $Y' \leftrightarrow s'$ $\Rightarrow$ If $\underset{dw_2}{ds} = d\theta\wedge s' + cs\wedge s'$

$[Y', Y] = cY + (bc + c')Y'$

$\underset{dw_3}{ds'} = (ds)' = d\theta\wedge s'' + c's\wedge s' + cs\wedge s''$ $\quad s'' = -s - bs'$

$= -d\theta\wedge s - bd\theta\wedge s' + (c' + cb)s\wedge s'$

$V \overset{\text{dual}}{\longleftrightarrow} W \Rightarrow$ If $dw_k = \sum\frac{1}{2}C^k_{ij}\, w_i\wedge w_j \Leftrightarrow [V_i, V_j] = -\sum_k C^k_{ij}\, V_k$

$\therefore$ Hori $= 0$.

$\therefore$ By defining $k := \vec{H}c' - \vec{H}'c$. Thm proved $\blacksquare$

$\Rightarrow \quad k := \vec{H}'c - \vec{H}c'$

Rmk). v : unique up to sgn $\Rightarrow \begin{cases} [\vec{H}, [\vec{H}, v]] = kv \\ [\vec{H}, [\vec{H}, -v]] = -kv = k(-v) \end{cases} \Rightarrow$ invariant to choice of v.

Rmk). k : intrinsic (only depend on v satisfying $L_v^2 s = -s + bL_v s$) $\Leftarrow s$ is intrinsic.

Topic 2). This Curvature $\overset{?}{=}$ classical Curvature

To this end).

①. formulate "geodesic of M" in a optimal control style

$$(\circledast\circledast\circledast)\quad\begin{cases} \dot{q} = u_1 X_1(q) + u_2 X_2(q) & |u| \le 1 \\ q(0) = q_0, \quad q(T) = q_T \\ \min\ T := \int_0^T 1\,dt \end{cases}$$

$\{X_1, X_2\}$: moving frame.

Then, Rie met G given this frame: $\begin{bmatrix} X_1^T X_1 & X_1^T X_2 \\ X_2^T X_1 & X_2^T X_2 \end{bmatrix}^{-1}$

②. Apply prev res.

$\circledast\circledast\circledast$'s Hamiltonian $H_u(\lambda) = \langle \lambda, f_u(q) \rangle$

$$\underset{\substack{\pi \\ \text{param by } (p, q) \\ \cap\ \ \cap \\ T_q^*M\ \ M}}{\lambda \in T^*M} \qquad = u_1 \langle p, X_1 \rangle_q + u_2 \langle p, X_2 \rangle_q$$

$\max_u H_u(p, q)$ is achieved when $u = \dfrac{(\langle p, X_1 \rangle, \langle p, X_2 \rangle)}{\| (\langle p, X_1 \rangle, \langle p, X_2 \rangle) \|}$ ($\langle p, X_1 \rangle = u_1$, $\langle p, X_2 \rangle = u_2$)

and in that case $H(\lambda) = \| \overbrace{(\langle p, X_1 \rangle, \langle p, X_2 \rangle)}^{\tilde{p}} \|$

As we did in the above, consider $H^{-1}(\{1\})$. i.e. $\|\tilde{p}\| = 1$

Then, $(\langle p, X_1 \rangle, \langle p, X_2 \rangle)$ can be parameterize by $(\cos\theta, \sin\theta) := V_\theta$

$$\Rightarrow \left.\begin{cases} \langle p, X_1 \rangle = \cos\theta \\ \langle p, X_2 \rangle = \sin\theta \end{cases}\right\} \underset{\substack{\uparrow \\ \left(\text{check by plug-in}\right)}}{\Rightarrow} \begin{aligned} p_1 &= \frac{\langle X^2, V' \rangle}{\det X} \\ p_2 &= \frac{-\langle X^1, V' \rangle}{\det} \end{aligned} \qquad \left((X_1\ X_2) = \underbrace{\begin{pmatrix} X_1^1 & X_2^1 \\ X_1^2 & X_2^2 \end{pmatrix}}_{=X} = \begin{matrix} X^1 \\ X^2 \end{matrix} \right)$$

$'$: diff by θ

$\therefore s|_H = p\,dq|_H = \frac{1}{\det X} (\langle X^2, V_\theta' \rangle dq_1 - \langle X^1, V_\theta' \rangle dq_2)$

$s'|_H = \frac{1}{\det X} (\frac{\partial}{\partial\theta} \langle X^2, V_\theta' \rangle dq_1 - \frac{\partial}{\partial\theta} \langle X^1, V_\theta' \rangle dq_2) = \frac{1}{\det X} (-\langle X^2, V \rangle dq_1 + \langle X^1, V \rangle dq_2)$

Lastly we need $\vec{H}$, c to compute k

To obtain, note $\vec{H} = Y - c\frac{\partial}{\partial\theta}$, $ds = d\theta \wedge s' + c s \wedge s' \Leftrightarrow ds|_H = c s \wedge s'|_H$

$$ds|_H = d\frac{\langle X^2, V' \rangle}{\det X} \wedge dq_1 - d\frac{\langle X^1, V' \rangle}{\det X} \wedge dq_2 \underset{\substack{\uparrow \\ \circledast.}}{=} \frac{1}{\det X} \Big(\frac{q_1'}{\det X} \frac{\partial}{\partial X_1} \det X - \frac{\partial}{\partial X_1} q_1'$$
$$+ \frac{q_2'}{\det X} \frac{\partial}{\partial X_2} \det X - \frac{\partial}{\partial X_2} q_2' \Big) dq_1 \wedge dq_2$$

$$s|_H \wedge s'|_H = \frac{1}{\det X^2} (\underbrace{\langle X^2, V' \rangle \langle X^1, V \rangle - \langle X^2, V \rangle \langle X^1, V' \rangle}_{X_1^1 X_2^2 - X_1^2 X_2^1 = \det X}) dq_1 \wedge dq_2$$

$\therefore s|_H \wedge s'|_H = \frac{1}{\det X} dq_1 \wedge dq_2$

$$\therefore\ ds|_H = \underbrace{\sum_{i=1}^{2} \left(\frac{g_i'}{\det X} \frac{\partial}{\partial x_i} \det X - \frac{\partial}{\partial x_i} g_i' \right)}_{=C} s|_H \wedge s'|_H .$$

$$\vec{H} = \left(\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, -\frac{\partial H}{\partial q_1}, -\frac{\partial H}{\partial q_2} \right)$$

$$H(p,q) = \| (\langle p, x_1\rangle, \langle p, x_2\rangle) \| = \sqrt{(p_1 X_1^1(q) + p_2 X_1^2(q))^2 + (p_1 X_2^1(q) + p_2 X_2^2(q))^2}$$

$$\Rightarrow K = \vec{H}'C - \vec{H}C' = \sum_{i,j=1}^{2} \Bigg[\frac{\frac{\partial}{\partial x_i}\det X}{\det X} \left(2\langle X^i, \frac{\partial}{\partial x_j} X^j \rangle + \langle X^j, \frac{\partial}{\partial x_j} X^i \rangle \right)$$

$$- \frac{\frac{\partial}{\partial x_i \partial x_j}\det X^{-1}}{\det X^{-1}} \langle X^j, X^i \rangle - \frac{\partial}{\partial x_i} \langle X^i, \frac{\partial}{\partial x_j} X^j \rangle \Bigg)$$

believe...

③. Relationship with Gaussian Curvature.

- From HW2 p4.

If $g = (dq_1)^2 + J^2(q)(dq_2)^2$

$$\Rightarrow K = -\frac{1}{J} \frac{\partial^2 J}{\partial (q_1)^2}$$

We show $\left\{ X_1 = \begin{pmatrix}1\\0\end{pmatrix},\ X_2 = \begin{pmatrix}0\\J^{-1}(q)\end{pmatrix} \right\}$: orthonormal frame on ③.
induce the same result.

$$G = \begin{pmatrix} X_1^T X_1 & X_1^T X_2 \\ X_2^T X_1 & X_2^T X_2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & J^2(q) \end{pmatrix} \rightsquigarrow g = (dq_1)^2 + J^2(q)(dq_2)^2$$

$$\det X = \begin{vmatrix} X_1^1 & X_2^1 \\ X_1^2 & X_2^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & J^{-1}(q) \end{vmatrix} = J^{-1}(q)$$

$X^1 = (1, 0)$

$X^2 = (0, J^{-1}(q))$

$-\frac{1}{J}\frac{\partial^2 J}{\partial q_1^2}$

$$\therefore K = J(q) \cdot \frac{\partial}{\partial q_1} J(q) \left(2\langle (1,0), \cancel{\frac{\partial}{\partial q_1}(1,0)}^{0} \rangle + \langle (1,0), \cancel{\frac{\partial}{\partial q_1}(1,0)}^{0} \rangle \right)$$

$$- J^{-1}(q) \frac{\partial^2}{\partial q_1^2} J(q) \underbrace{\langle (1,0), (1,0) \rangle}_{=1} - \frac{\partial}{\partial q_1} \langle (1,0), \cancel{\frac{\partial}{\partial q_1}(1,0)}_{0} \rangle \Big)$$

$$+ J(q) \frac{\partial}{\partial q_2} J(q) \left(2\langle (1,0), \cancel{\frac{\partial}{\partial q_2}(0, J^{-1}(q))}^{0} \rangle + \langle (1,0), \cancel{\frac{\partial}{\partial q_2}(1,0)}^{0} \rangle \right)$$

$$- J^{-1}(q) \frac{\partial^2}{\partial q_1 \partial q_2} J(q) \left(\cancel{\langle X^2, X^1 \rangle}_{0} - \frac{\partial}{\partial q_1} \langle (1,0), \cancel{\frac{\partial}{\partial q_2}(0, J^{-1}(q))}_{0} \rangle \right)$$

(2.1) : same

(2.2) $\quad 3 J J_2$

$$J(q) \frac{\partial}{\partial q_2} J(q) \left(2 \langle (0, J^{-1}), \frac{\partial}{\partial q_2}(0, J^{-1}) \rangle + \langle (0, J^{-1}), \frac{\partial}{\partial q_2}(0, J^{-1}) \rangle \right)$$

$$- J^{-1} \cdot \frac{\partial^2 J}{\partial q_2^2} \langle (0, J^{-1}), (0, J^{-1}) \rangle - \frac{\partial}{\partial q_2} \langle (0, J^{-1}), (0, \frac{\partial J^{-1}}{\partial q_2}) \rangle$$

$$J \cdot J_2 \cdot 3 J^{-1} \underbrace{(J^{-1})_2}_{(J_2)(J^{-1})} - J^{-1} \cdot J_{22} \cdot J^{-2} - \big(J^{-1} \cdot \underbrace{(J^{-1})_2}_{(J_2)(J^{-1})} \big)_2$$

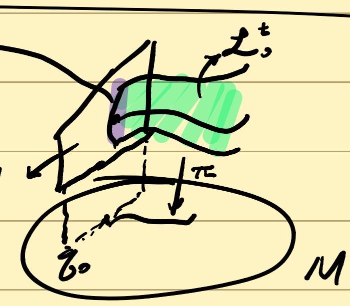
$[(J_2)(J^{-1})]^2 + J^{-1} J_{22}(J^{-1})$

$J_2(J^{-1})$

$$= -\frac{1}{J} \frac{\partial^2 J}{\partial q_1^2} \ : \ \text{coincide.}$$

Jacobi eq. $\quad H_q$

$\mathcal{L}_0^t := \underbrace{e^{t\vec{H}}}_{\text{flow of Hamiltonian field.}} (H_q)$: Curve in H_q $\quad$ ($\mathcal{L}_0^t \to$ 2-dim) $\quad T_q^*M$



def). $q(t)$ is conjugate to q_0 $\Leftrightarrow$ q : critical value of $\pi : \mathcal{L}_0^t \to M$
(time t " to 0) $\quad$ canonical projection

$\Updownarrow$ $\quad$ meaning). Curve collapse. (ending up being same)

Then, known that $(e_*^{t\vec{H}} v)(\lambda) \in \operatorname{span}(\vec{H}(\lambda), v(\lambda))$ $\quad$ $\vec{H}, v$: from above.

$(e_*^{t\vec{H}} v \in H.$ ($\because$ subspace : $\operatorname{span}\{\vec{H}, v\}$) if conjugate

$\therefore$ $(e_*^{t\vec{H}} v) = \alpha(t) v + \beta(t) \vec{H} + \gamma(t) [\vec{H}, v]$ $\quad$ "

Lemma 23.3). $\quad$ V_j : moving frame. then

$$(e_*^{tX}) V_j = \sum \gamma_j^i(t) V_i$$

where $\quad \Gamma(t) := (\gamma_j^i)$: $\exists!$ Sol of ODE IVP

$$\begin{cases} \Gamma'(t) = A(t)\Gamma(t), \\ \Gamma(0) = (V_1, \dots V_n) \end{cases}$$

$A(t) = a_j^i(t)$, $\quad a_j^i(t)$: $[X, V_j] = \sum_i a_j^i(t) V_j$

pf). skip

plug-in). $e_*^{t\vec{H}} v = \alpha(t) v + \beta(t) \vec{H} + \gamma(t) [\vec{H}, v] \Rightarrow$

$[\vec{H}, v] = \overset{0}{\overset{\shortparallel}{a_1^1}} V_1 + \overset{1}{\overset{\shortparallel}{a_1^2}} V_2 + \overset{0}{\overset{\shortparallel}{a_1^3}} V_3$

$[\vec{H}, [\vec{H}, v]] = a_2^1 V_1 + a_2^2 V_2 + a_2^3 V_3$

$[\vec{H}, \vec{H}] = 0$ $\quad \hookrightarrow = -k$ $\quad \hookrightarrow a_3^i = 0$

$\therefore\ A(t) = \begin{pmatrix} 0 & -k(t) & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dot{P}(t) = e^{t\vec{H}} \begin{pmatrix} 0 & -k & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Gamma(t)$ ... Called Cauchy Problem.

$\Rightarrow \left\{ \begin{array}{l} \alpha' = -k(t)\beta(t) \\ \beta' = \alpha(t) \\ \gamma(t) = 0 \end{array} \right\} \Rightarrow \left[\underbrace{\beta''(t) + k(t)\beta(t) = 0}_{\beta(0) = \beta(t) = 0} \right] \Rightarrow$ t : conjugate time

$\Leftrightarrow \left. \begin{array}{l} x''(t) + k(t)x(t) = 0 \\ x(0) = x(t) = 0 \end{array} \right] \Rightarrow$ Jacobi eq.

$\exists$ sol.

Thm 23.5). Using Strum Comparison Thm of 2^{nd} order ODE.

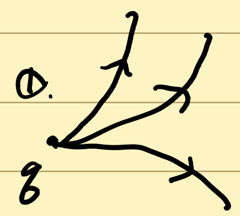
$K(t) = K(\lambda_t) = K(\,(p, q_t)\,)$ $(\text{or} = K(\theta, q).)$

① $K \le 0 \Rightarrow$ No conjuagate point.

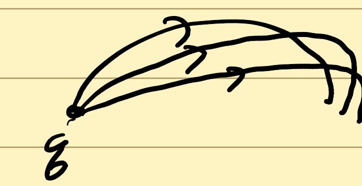
② $K < C^2$ for some $C > 0 \Rightarrow$ No Conjuagate $[0, \frac{\pi}{c}]$

③ $K \ge C^2$: $\exists$ Conjuagate on $[0, \frac{\pi}{c}]$

case of ①. $K < 0$



Case of ③ $K > 0$



Its flow $B_t := \underset{\text{exponential map.}}{\exp} \int_0^T \vec{b_\tau}\, d\tau$.

Let $\Pi_0 := T_{\lambda_0} T^*_{q_0} M \le \Sigma$, Vertical subsp., $C_t \le \Pi_0$ → Constant vertical solution (21.87)

$C_t \subset B_t(\Pi_0) \cap \Pi_0$ ($\because B_t$ always contain constant vertical)

if $\ne$ 〃, t is called a conjuagate time.

- Vertical Euler field $E \in Vec(T^*M)$

flow $\lambda \circ e^{tE} = e^t \cdot \lambda$ for $\lambda \in T^*M$.

If (p, q): Coordi of $T^*M. \Rightarrow E = p\frac{\partial}{\partial p}$

$\{V_1, V_2, V_3, E\}$: Basis of $\overset{4\text{-dim}}{T_\lambda(T^*M)}$ $\lambda \in \mathcal{H}$

$V_1 \perp\!\!\!\perp E$

$\Pi_0 = span(\,V_1(\lambda_0), E(\lambda_0)\,)$

for C_t : evaluate action of the flow B_t
(to get const)

known → $B_t(\eta) = (P_t^*)_* \, e^{t\vec{H}}_*(\eta)$

$\{ d\theta, s, s' \}$: s'

$ds = a \cancel{d\theta \wedge s} + \underset{=1}{b}\, d\theta \wedge s' + c\, \underbrace{s \wedge s'}_{d_y s}$ $\underbrace{L_v s}$

$s = a\, d\theta + \underbrace{b\, dx + c\, dy}$

$ds = \frac{\partial}{\partial\theta} b\, d\theta \wedge dx + \frac{\partial}{\partial\theta} c\, d\theta \wedge dy =$

$d\theta \wedge \left(\frac{\partial}{\partial\theta} a\, d\theta + \underbrace{\frac{\partial}{\partial\theta} b\, dx + \frac{\partial}{\partial\theta} c\, dy}_{L_v s} \right) +$

$+ \underbrace{d' s}_{\text{only derivatives w.r.t } x \,\&\, y}$