§13.4. Groenewold No-Go Thm.

Goal). Show 'nice' quantization does not exist.

* Background.
Let $x \in \mathbb{R}^d$: position vector & $p \in \mathbb{R}^d$: momentum vector

- Poisson Bracket
for $f, g \in C^\infty(M) \Rightarrow f, g$ stand for observables.
(M: symplectic mfd (phase space))

$$\{f, g\} = \sum_i^d \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right)$$

- Quantum extension
let $\varphi \in L^2(\mathbb{R}^d; \mathbb{C})$: State vector (wave fn)

$$\left.\begin{aligned} \hat{x} &: \mathrm{Dom}(\hat{x}) \subseteq L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \quad : \text{position operator} \\ &\qquad \varphi(x) \mapsto x\,\varphi(x) \\ \hat{p} &: \mathrm{Dom}(\hat{p}) \subseteq L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \quad : \text{momentum operator.} \\ &\qquad \varphi(x) \mapsto -i\hbar \frac{\partial}{\partial x} \varphi(x) \end{aligned}\right\} \quad (0).$$

is a 'Natural' quantization of position and momentum
(Natural from either natural properties we expect from quantization (See lec 20)
or physical property called $\underbrace{\text{'de Broglie hypothesis'}}$)
↳ $p_\varphi = \hbar \, \mathrm{freq}(\varphi)$ for φ: wave fn

Then, simple calculations lead to
(1). $[\hat{x}, \hat{p}] = \frac{1}{i\hbar}\{x, p\}$ $\Rightarrow$ Natural quantized operation corresponding to Poisson Bracket is Commutator.

- Generalization: Weyl quantization

Notations). $\mathcal{P}$: A set of all polynomials in $(x, p) \in \mathbb{R}^{2d}$
$\mathcal{P}_k$: A set of k-degree 'homogeneous' polynomial of (x, p)
$\mathcal{P}_{\leq k}$: A set of polynomials of at most k degrees.
$D(\mathbb{R}^d)$: A space of differential operators on $\mathbb{R}^d$ with polynomial coefficients.
(e.g. $\sum_n f_n(x) \left(\frac{\partial}{\partial x}\right)^n$ n: multi-indices)
f_n : polynomials.

- Weyl quantization

for $f \in L^2(\mathbb{R}^{2d})$, $Q_{weyl}(f) := \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f\left(\frac{x+y}{2}, p\right) e^{-i\frac{(y-x)\cdot p}{\hbar}} \, dp \, dy$

$Q_{weyl} : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$ ← In precise version operator on $\mathcal{S}(\mathbb{R}^d)$ where $\mathcal{S}(\mathbb{R}^d)$: Schwartz space but it can be extended to L^2 by limiting argument. ($\mathcal{S}(\mathbb{R}^d)$: dense in $L^2(\mathbb{R}^d)$)

$$f \mapsto \left(\varphi \mapsto \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\left(\frac{x+y}{2}, p\right) e^{-i(y-x)\cdot p/\hbar} \cdot \varphi(y) \, dp \, dy \right)$$

Rmk). f produces a kernel $k_f(x, y) := \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbb{R}^d} f\left(\frac{x+y}{2}, p\right) e^{-i\frac{(y-x)}{\hbar}\cdot p} \, dp$, and
Q is a kernel integral operator of k_f & φ. i.e. $Q(f) = \int k_f(x, y)\varphi(y)\,dy$

Called Moyal product.

formula). $Q(f)Q(g) = Q(f*g)$ where $(f*g)(x,p) = \left(\frac{1}{2\pi}\right)^d \iint_{\mathbb{R}^{2d}} e^{-i\hbar\left(\frac{x\cdot p' - p\cdot x'}{2}\right)} \hat{f}(x-x', p-p')\hat{g}(x',p')\,dx'dp'$

$$\Rightarrow \begin{cases} Q(x_j^n) = \hat{x}_j^{\,n} \\ Q(p_j^n) = \hat{p}_j^{\,n} \end{cases} \quad Q(x^j p^k) = \frac{1}{(j+k)!}\sum_{\sigma\in S_{j+k}} X_{\sigma(1)}\cdots X_{\sigma(j)} P_{\sigma(j+1)}\cdots P_{\sigma(j+k)} \qquad \circledast$$

- Weyl quantization is natural extension of (1). (Prop 13.11.).
  For $f\in P_{\le 2}$, $g\in P$.

$$Q(\{f,g\}) = \frac{1}{i\hbar}[Q(f), Q(g)] \quad \text{--} \quad (2).$$

Our Main Q). Does (2) can be extended to $f\in P$? (X).

Thm). Groenewold's No-Go Thm. (Thm 13.13)
$\nexists Q : P_{\le 4} \to D(\mathbb{R}^d)$, linear s.t.

①. $Q(1) = I$

②. $Q(x_j) = \hat{x}_j$, $Q(p_j) = \hat{p}_j$ where $\hat{x}_j, \hat{p}_j$: defined as (0).

③. $Q(\{f,g\}) = \frac{1}{i\hbar}[Q(f), Q(g)]$ for $\forall f,g\in P_{\le 3}$ (to make $\underbrace{\{f,g\}}_{(\deg(f)-1)+(\deg(g)-1)}\in P_{\le 4}$)

Rmk). Meaning of Thm $\Rightarrow$ No natural quantization satisfying the correspondence $\{\cdot,\cdot\}\approx[\cdot,\cdot]$

$\Rightarrow$ Sol). Replace Poisson Bracket into Moyal Bracket $\{\{\cdot,\cdot\}\}$.

$$\{\{f,g\}\} := \frac{1}{i\hbar}(f*g - g*f) = \{f,g\} + o(\hbar)$$

$\exists$ Quantization $\{\{\cdot,\cdot\}\}\approx[\cdot,\cdot]$.

pf strategy).

(i). We know at least for $f\in P_{\le 2}$, $g\in P_{\le 3}$ Q_{Weyl} satisfies ①②③ $\Rightarrow$ show indeed it is an unique option.

(ii). Then we show $\exists f = \{g,h\} = \{g',h'\}\in P_4$ s.t. $Q_W(f)$ is not well-defined as $Q(\{g,h\}) \ne Q(\{g',h'\})$

- For proof, we need some Lemmas.

Lem 13.14. If $A\in D(\mathbb{R}^d)$ has a form $A := \sum_k f_k(x)\left(\frac{\partial}{\partial x}\right)^k$ with $f_k \not\equiv 0$ only finitely many.
Then. $A \equiv 0$-operator on $C_c^\infty(\mathbb{R}^d) \Leftrightarrow \forall k\ f_k\equiv 0$

pf). let $k = (k_1, \dots, k_d)$

Assume the opposite : $\exists k$ s.t. $f_k\not\equiv 0$, while $A\equiv 0$-operator.

By assumption, $\exists k^0$ s.t. $f_{k^0}\not\equiv 0$ & $|k^0| = \min\{|k| \mid f_k\not\equiv 0\}$.

Now, take $g(x) = x^{k^0}$ in $U(0)$

$Ag = f_{k^0}(x) \ne 0$. (*). $\square$(Lem).

To remove residual parts besides Q_W

Lem 13.15. If $A\in D(\mathbb{R}^d)$, A commutes with X_j and P_j $\forall j = 1,\dots,d$ $\Rightarrow A = cI$ for some $c\in\mathbb{C}$.

pf).

1) Note that $\left(\frac{\partial}{\partial x_j}\right)^k (x_j g(x)) = k\left(\frac{\partial}{\partial x_j}\right)^{k-1} g(x) + x_j\left(\frac{\partial}{\partial x_j}\right)^k g(x)$ for fixed j

- By induction

$k=1 \Rightarrow \frac{\partial}{\partial x_j} x_j g(x) = g(x) + x_j\frac{\partial}{\partial x_j} g(x)$ (0)

Suppose it holds for $1,\dots,k \Rightarrow \left(\frac{\partial}{\partial x_j}\right)^{k+1}(x_j g(x)) = \left(\frac{\partial}{\partial x_j}\right)\left(\left(\frac{\partial}{\partial x_j}\right)^k (x_j g(x))\right) = \frac{\partial}{\partial x_j}\left(k\left(\frac{\partial}{\partial x_j}\right)^{k-1} g(x) + x_j\left(\frac{\partial}{\partial x_j}\right)^k g(x)\right)$

$= k\left(\frac{\partial}{\partial x_j}\right)^k g(x) + \left(\frac{\partial}{\partial x_j}\right)^k g(x) + x_j\left(\frac{\partial}{\partial x_j}\right)^{k+1} g(x) = (k+1)\left(\frac{\partial}{\partial x_j}\right)^k g(x) + x_j\left(\frac{\partial}{\partial x_j}\right)^{k+1} g(x)$

For $\forall \varphi \in L^2(\mathbb{R}^d)$

2). $[f(x)(\frac{\partial}{\partial x})^k, \hat{x}_j](\varphi) = f(x)(\frac{\partial}{\partial x})^k(\hat{x}_j(\varphi)) - \hat{x}_j(f(x)(\frac{\partial}{\partial x})^k(\varphi))$

$$= f(x)\left(\underbrace{(\tfrac{\partial}{\partial x})^k(x_j\varphi(x))} - x_j(\tfrac{\partial}{\partial x})^k(\varphi)\right)$$

$$\left(\begin{aligned} &= \frac{\partial^{|k|}}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_j^{k_j}\cdots\partial x_d^{k_d}}(x_j\varphi(x)) \\ &= (\tfrac{\partial}{\partial x_{-j}})^{k-k_j}\underbrace{(\tfrac{\partial}{\partial x_j})^{k_j}(x_j\varphi(x))}_{= k_j(\frac{\partial}{\partial x_j})^{k_j-1}(\varphi) + x_j(\frac{\partial}{\partial x_j})^{k_j}(\varphi)} \\ &= k_j(\tfrac{\partial}{\partial x})^{k-e_j}(\varphi) + x_j(\tfrac{\partial}{\partial x})^k(\varphi) \end{aligned}\right)$$

$$= k_j f(x)(\tfrac{\partial}{\partial x})^{k-e_j}(\varphi)$$

$\therefore [f(x)(\frac{\partial}{\partial x})^k, \hat{x}_j] = k_j f(x)(\frac{\partial}{\partial x})^{k-e_j}$

Now, take A satisfying the assumption (commute with x_j & p_j)

Let $\deg(A) = M$.

·claim). $M=0$.

Pf by contra) dicton Suppose $M>0 \Rightarrow \exists k_0$ s.t. $|k_0|=M$ & $A(\cdot) = \underbrace{f_{k_0}(x)}_{\neq 0}\cdot(\frac{\partial}{\partial x})^{k_0} + \alpha(\cdot)$ denotes other terms.

since $|k_0|=M>0 \Rightarrow \exists j$ s.t. $k_j>0$.

Then, $\underset{\uparrow \text{ assumption}}{0 =} [A, x_j] = [f_{k_0}(x)(\frac{\partial}{\partial x})^{k_0} + \alpha, \hat{x}_j] = \underbrace{(k_0)_j f_{k_0}(x)(\frac{\partial}{\partial x})^{k_0-e_j}}_{\neq 0.} + \underbrace{[\alpha, \hat{x}_j]}_{k-e_j \text{ terms, but } k\neq k_0}$ $\Rightarrow \exists$ non-zero coefficient.

$\Rightarrow$ does not coincide with the former term.

$\underset{\uparrow \text{ Lem 13.14.}}{\Rightarrow} [A, x_j] \neq 0$. $(\neq)$ with A: commute with x_j

□ (claim)

$\therefore$ $M=0 \Rightarrow A(x) = f(x)$ (0-degree differential $= \mathcal{C}^\infty(\mathbb{R}^d)$)

Now, from $[A, \hat{p}_j]=0$ (assumption) $\Rightarrow \forall \varphi \in L^2(\mathbb{R}^d)$

$$0 = [f(x), -i\hbar\tfrac{\partial}{\partial x_j}](\varphi) = -i\hbar[f(x), \tfrac{\partial}{\partial x_j}](\varphi)$$

$$= -i\hbar\left(f(x)\frac{\partial\varphi}{\partial x_j} - \underbrace{\frac{\partial}{\partial x_j}(f(x)\varphi)}_{= \frac{\partial f}{\partial x_j}\varphi + f(x)\frac{\partial\varphi}{\partial x_j}}\right)$$

$$= i\hbar\frac{\partial f}{\partial x_j}\cdot\varphi$$

$\therefore \frac{\partial f}{\partial x_j} = 0 \Leftrightarrow A(x) = \text{const w.r.t. } x \Rightarrow A(x) = cI$. ☑ Lem 13.15.

Lem 13.16. $\forall f \in \mathcal{P}_2$ $\exists g_1 \cdots g_j, h_1 \cdots h_j \in \mathcal{P}_2$ s.t. $f = \sum_{i=1}^j \{g_i, h_i\}$.

$f' \in \mathcal{P}_3$ $\exists g_1' \cdots g_k' \in \mathcal{P}_3$, $h_1' \cdots h_k' \in \mathcal{P}_2$ s.t. $f' = \sum_i^k \{g_i', h_i'\}$. (expressing $\mathcal{P}_2$ elements by Poisson Bracket of $\mathcal{P}_2$'s)

pf). Let $g_1(x, p) := \sum_{j=1}^n x_j p_j$

$\forall x^i p^k = x_1^{i_1}\cdots x_d^{i_d} p_1^{k_1}\cdots p_d^{k_d}$

$\{g_1, x^i p^k\} = \sum_j^n \{x_j p_j, x_1^{i_1}\cdots x_j^{i_j}\cdots x_d^{i_d} p_1^{k_1}\cdots p_j^{k_j}\cdots p_d^{k_d}\}$

$$= \sum_j \sum_l \left(\underbrace{\frac{\partial x_j p_j}{\partial x_l}}_{\delta_{jl}p_j}\cdot\underbrace{\frac{\partial x^i p_1^{k_1}\cdots p_d^{k_d}}{\partial p_l}}_{k_l x^i p^{k-e_l}} - \underbrace{\frac{\partial x_j p_j}{\partial p_l}}_{\delta_{jl}x_j}\cdot\underbrace{\frac{\partial x_1^{i_1}\cdots x_d^{i_d} p^k}{\partial x_l}}_{i_l x^{i-e_l} p^k}\right)$$

$$= \sum_j\left(k_j\underbrace{p_j x^i p^{k-e_j}}_{p^k} - i_j\underbrace{x_j x^{i-e_j}}_{x^i}p^k\right) = \sum_j(k_j - i_j)x^i p^k = (|k|-|i|)x^i p^k$$

Since $\forall f \in \mathcal{P}_n$ $f(x,p) = \Sigma_{|i|+|k|\le n} a_{ik} x^i p^k = \Sigma_{|i|+|k|\le n} \underbrace{\frac{a_{ik}}{|k|-|i|}}_{:=b_{ik}} \underbrace{(|k|-|i|) x^i p^k}_{\{g_1, x^i p^k\}} + \Sigma_{|i|\ne|k|} a_{ik} x^i p^k$

$$= \Sigma \{ \underset{\underset{(h_\ell)}{\mathcal{P}_2 \,\cap}}{b_{ik} g_1}, \underset{\underset{(g_\ell)}{\mathcal{P}_n\,\cap}}{x^i p^k} \} + \underbrace{\Sigma_{|i|\ne|k|} a_{ik} x^i p^k}_{\text{in } n=3 \to \text{Vanish.}}$$

In case of $|i|=|k|=1$, $x^i p^k = x_j p_j = \{\underset{\underset{\in \mathcal{P}_2}{g_\ell}}{\tfrac{1}{2} x_j^2}, \underset{\underset{\in \mathcal{P}_2}{h_\ell}}{\tfrac{1}{2} p_j^2}\}$

$\square$ (Lem 13.16)

Lem 13.17. (Uniqueness of quantization)
$\mathcal{Q}: \mathcal{P}_{\le 3} \to D(\mathbb{R}^d)$ satisfying ①.②.③ $\Rightarrow \mathcal{Q} = \mathcal{Q}_w$

pf). By the construction of Weyl quantization, we have $\underbrace{\mathcal{Q}|_{\mathcal{P}_{\le 1}} = \mathcal{Q}_w}_{⊛⊛}$ (See Prop 13.11.)

So consider a case $f \in \mathcal{P}_2$
Write $\mathcal{Q}(f) = \mathcal{Q}_w(f) + A_f$.
$\forall g \in \mathcal{P}_{\le 1}$, from ①.②.③ we have

$$\mathcal{Q}(\{f,g\}) \overset{③}{=} \frac{1}{i\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)] \overset{\text{assumption}}{=} \frac{1}{i\hbar}[\mathcal{Q}_w(f) + A_f, \mathcal{Q}_w(g)]$$
$$= \frac{1}{i\hbar}[\mathcal{Q}_w(f), \mathcal{Q}_w(g)] + \frac{1}{i\hbar}[A_f, \mathcal{Q}_w(g)]$$
$$\overset{③}{=} \mathcal{Q}_w(\{f,g\}) + \frac{1}{i\hbar}[A_f, \mathcal{Q}_w(g)]$$
$$\overset{⊛⊛}{=} \mathcal{Q}(\{f,g\}) + \frac{1}{i\hbar}[A_f, \mathcal{Q}_w(g)] \qquad ⊛⊛⊛$$

$f \in \mathcal{P}_2, g \in \mathcal{P}_{\le 1}$
$$\Rightarrow \{f,g\} = \Sigma\Big(\underbrace{\frac{\partial f}{\partial x}}_{\in \mathcal{P}_{\le 1}} \cdot \underbrace{\frac{\partial g}{\partial p}}_{\in \mathcal{P}_{\le 0}} - \underbrace{\frac{\partial f}{\partial p}}_{\in \mathcal{P}_{\le 1}} \underbrace{\frac{\partial g}{\partial x}}_{\in \mathcal{P}_{\le 0}}\Big) \in \mathcal{P}_{\le 1}$$

$\therefore \frac{1}{i\hbar}[A_f, \mathcal{Q}_w(g)] = 0$.

Since the choice of g is arbitrary, taking $g = x_j$ and $p_j \Rightarrow A_f$ commutes with $\hat{x}_j, \hat{p}_j$ $\forall j$

$$\underset{\text{Lem 13.15.}}{\Rightarrow} A_f = c_f I.$$

$\therefore$ For $f \in \mathcal{P}_2$, $\mathcal{Q}(f) = \mathcal{Q}_w(f) + c_f I$.
$\therefore$ Let $f, h \in \mathcal{P}_2$

$$\mathcal{Q}(\{f,h\}) \underset{③}{=} \frac{1}{i\hbar}[\mathcal{Q}(f), \mathcal{Q}(h)] = \frac{1}{i\hbar}[\mathcal{Q}_w(f) + c_f I, \mathcal{Q}_w(h) + c_h I]$$
$$\overset{\text{bilinearity \& } cI \text{ commutes}}{=} \frac{1}{i\hbar}[\mathcal{Q}_w(f), \mathcal{Q}_w(h)] \underset{③}{=} \mathcal{Q}_w(\{f,h\})$$

$\therefore \mathcal{Q}(\{f,h\}) = \mathcal{Q}_w(\{f,h\})$ $\forall f, h \in \mathcal{P}_2$

Now, from Lem 13.16. we have $\forall g \in \mathcal{P}_2 \quad g = \sum \{\overset{\mathcal{P}_2 \ni}{f_i}, \overset{\mathcal{P}_2 \ni}{h_i}\}$

$\therefore$ By linearity of Q. $Q(g) = Q_w(g) \quad \forall g \in \mathcal{P}_2 \cup \underbrace{\mathcal{P}_{\le 1}}_{\text{already have}} = \mathcal{P}_{\le 2}$
& $c_f = 0 \quad \forall f \in \mathcal{P}_2$

repeat the above

For $f \in \mathcal{P}_3$ we again use same trick

$Q(f) = Q_w(f) + B_f$

Take $g \in \mathcal{P}_{\le 1} \Rightarrow \{f.g\} \in \mathcal{P}_{\le 2}$ with

By same logic in ⊛, we have $B_f = c_f I$ again

Then, take $h \in \mathcal{P}_2$. → to use Lem 13.16.

$Q(\{f.h\}) = \frac{1}{i\hbar}[Q_w(f) + c_f I \,.\, Q_w(h)] = Q_w(\{f.g\})$ again.

$\Rightarrow$ Use Lem 13.16 with $\mathcal{P}_3 = \sum\{\mathcal{P}_3 . \mathcal{P}_2\} \Rightarrow Q(f) = Q_w(f)$ in $f \in \mathcal{P}_3 \cup \mathcal{P}_{\le 2} = \mathcal{P}_{\le 3}$.

▨ Lem 17.

- proof of main Thm.

Proof by contradiction.
Let assume $\exists$ such Q.
Take $f(x,p) = x_1^2 p_1^2 \in \mathcal{P}_4$
we observe the following: $x_1^2 p_1^2 = \frac{1}{9}\{x_1^3, p_1^3\} = \frac{1}{3}\{x_1^2 p_1, x_1 p_1^2\}$

$$\therefore Q(x_1^2 p_1^2) = \frac{1}{9i\hbar} \underbrace{Q(\{x_1^3, p_1^3\})}_{\substack{\| \\ [Q_w(x_1^3), Q_w(p_1^3)]}} = \frac{1}{3i\hbar} \underbrace{Q(\{x_1^2 p_1, x_1 p_1^2\})}_{\substack{\| \\ [Q_w(x_1^2 p_1), Q_w(x_1 p_1^2)]}}$$

Now, take $\varphi = 1$ (constant fn)

$$[Q_w(x_1^3) \,.\, Q_w(p_1^3)](1) \overset{⊛}{=} \underbrace{\hat{x}_1^3 \hat{p}_1^3(1)}_{\substack{= x_1^3 (-i\hbar)^3 (\frac{\partial}{\partial x_1})^3 (1) \\ = 0}} - \underbrace{\hat{p}_1^3 \hat{x}_1^3(1)}_{\substack{= (-i\hbar)^3 (\frac{\partial}{\partial x_1})^3 x_1^3 \\ = (-i\hbar)^3 6}} = -(i\hbar)^3 6$$

Using ⊛ ⊕ $\hat{p}\hat{x}\hat{p} = \frac{1}{2}(\hat{x}\hat{p}^2 + \hat{p}^2\hat{x})$, we have
$Q_w(x_1 p_1^2) = \frac{1}{2}(\hat{x}_1 \hat{p}_1^2 + \hat{p}_1^2 \hat{x}_1)$
$Q_w(x_1^2 p_1) = \frac{1}{2}(\hat{x}_1^2 \hat{p}_1 + \hat{p}_1 \hat{x}_1^2)$ (DIY)

$$[Q_w(x_1^2 p_1), Q_w(x_1 p_1^2)](1) = \frac{1}{4}\Big((\hat{x}_1^2 \hat{p}_1 + \hat{p}_1 \hat{x}_1^2)(\hat{x}_1 \hat{p}_1^2 + \hat{p}_1^2 \hat{x}_1)(1) - (\hat{x}_1 \hat{p}_1^2 + \hat{p}_1^2 \hat{x}_1)(\hat{x}_1^2 \hat{p}_1 + \hat{p}_1 \hat{x}_1^2)(1)\Big)$$

$$= \frac{1}{4}\Big(\hat{x}_1^2 \hat{p}_1 \hat{x}_1 \underbrace{\hat{p}_1^2(1)}_{=0} + \hat{p}_1 \hat{x}_1^3 \underbrace{\hat{p}_1^2(1)}_{=0} + \hat{x}_1^2 \underbrace{\hat{p}_1^3 \hat{x}_1(1)}_{(\frac{\partial}{\partial x_1})^3 x_1 = 0} + \hat{p}_1 \hat{x}_1^2 \underbrace{\hat{p}_1^2 \hat{x}_1(1)}_{(\frac{\partial}{\partial x_1})^2 x_1 = 0}$$
$$- \hat{x}_1 \hat{p}_1^2 \hat{x}_1^2 \underbrace{\hat{p}_1(1)}_{=0} - \hat{p}_1^2 \hat{x}_1^3 \underbrace{\hat{p}_1(1)}_{=0} - \hat{x}_1 \underbrace{\hat{p}_1^3 \hat{x}_1^2(1)}_{(\frac{\partial}{\partial x_1})^3 x_1^2 = 0} - \underbrace{\hat{p}_1^2 \hat{x}_1 \hat{p}_1 \hat{x}_1^2(1)}_{= (-i\hbar)^3 (\frac{\partial}{\partial x_1})^2 x_1 \cdot \underbrace{\frac{\partial}{\partial x_1} x_1^2}_{2x_1}}\Big)$$
$$= (-i\hbar)^3 \cdot 1$$

$\Rightarrow Q(x_1^2 p_1^2) = -(i\hbar)^2 \frac{2}{3} = -(i\hbar)\frac{1}{3}$ (✱)

▨ Thm.